No. XXII.

Obfervations on the Figure of the Earth. By JOSEPH CLAY, M. A. P. S.

THE fubject of this paper was fuggefted to me by a perufal of the " Studies of Nature," by Bernardin de St. Pierre. The pofitive manner in which that author afferts that the earth is a prolate fpheroid, the arrogance with which he challenges refutation, and above all the erroneous theories which he has built on this affertion, feem to require all doubts to be removed by a mathematical demonftration. It is known that degrees of latitude increafe in length as we approach to the poles. Upon this ground, St. Pierre places his principal argument which in fubftance is that if two lines diverging from the centre of an ellipfis, intercept a part of the curve, the further that part is from the centre, the longer will it be; and converfely, as the arch of one degree is longer near the pole than an arch of one degree near the equator, the axis muft be longer than the equatorial diameter. His error arifes from fuppofing, that degrees of latitude are meafured by the angles of femi-diameters of the meridian. This is not the cafe. The only mode of determining the latitude is by obferving the altitude of the heavenly bodies, either by the mural quadrant or fector or by Hadley's octant. Suppofing the fun to be the body altitude of which is taken, and fuppofing it to be in the equator and on the meridian, the complement of its altitude is equal to the latitude of the place of obfervation. The parallax of the fun is fo fmall, that rays of light coming from it may without fenfible error be confidered as coming in parallel lines; this being premifed, let

two

two right lines *h l o* (Fig. 1.) and HLO reprefent two tangents to the fame meridian; and let *ſ l* and SL reprefent two rays, parallel to each other, and to the common diameter of the meridian of the place and the equator; the angles *ſ l h* and SLH will be the altitude of the fun at *l* and L as taken with Hadley's octant. Draw *z l m* and ZLM perpendicular to the refpective tangents through *l* and L and meeting each other in M, then will the angles *ſ l z* and SLZ be the latitudes of *l* and L. Hence it appears that the latitude of a place is meafured by the angle formed by the common diameter of the meridian and equator, and a perpendicular to the horizon of the place; for the lines *ſ l* and SL are parallel to the common diameter of the equator and meridian (by conftruction). Produce SL to T. The angle ST *l* is equal to the angle *ſ l z*, and confequently to the latitude of *l* and the angle TLM (equal to SLZ) is equal to the latitude of L. The angle ST *l* is equal to the angles TLM and LMT taken together and confequently the angle LMT is equal to the difference between the two angles ST *l* and TLM, equal to the difference between the latitudes of the two places. That is, the difference of latitude between two places on the fame meridian, is meafured by the angle formed by the perpendiculars to the two horizons.*

By all the obfervations made at Greenwich and elfewhere, the altitudes of the heavenly bodies as obferved with the mural and plummet quadrants agree with thofe taken with the reflecting or Hadley's octant.† Now let ABDE be an ellipfis (Fig. 2.) and HLO a tangent, ZLT a perpendicular to that tangent *ſ*L a ray of light (the fun being in the equator and on the meridian) *ſ*LZ is the

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* In this demonftration nothing, which has been before demonftrated, is, on that account alone, omitted.

† This part of the demonftration is neceffarily experimental, not mathematical.

the ſun's zenith diſtance, and conſequently equal to the latitude of the place. It is evident that bodies near the ſurface of the earth, are not attracted in lines paſſing through the earth's centre; but in lines perpendicular to the horizon; for if it were otherwiſe a plummet would hang in the direction QLC (paſſing through the centre of the ellipſis) and the latitude of the place would in that caſe be equal to the angle ſLQ: but this angle never would, except under the poles and at the equator, coincide with the angle ſLZ. It is plain, therefore, that the difference of latitude cannot, with any inſtrument, be meaſured by the angles between lines meeting in the earth's centre.

But as the difference of latitude is meaſured by the angle formed between the perpendiculars to the two horizons, it follows that the nearer the curve of the meridian approaches to a right line, the longer muſt the part of the arch be which ſubtends any given angle.

Beſides it is evident, that were the earth a plane, and of its actual diameter, no ſenſible difference would be obſerved in the ſun's altitude on any part of its ſurface, and of courſe the nearer the earth approaches to a plane, the leſs will be the difference of altitudes obſerved by two perſons at any given diſtance, and conſequently the degrees of latitude muſt be longer as the earth is flatter.

Independent of theſe circumſtances, let ABDE be an ellipſis of which AD and BE are the axes and C the centre. Make CF equal to AC. Draw AF which produce to G. Biſect AG in K. Draw KC which produce to L and R. Through L draw HLO parallel to AG and cutting AD and BE produced in O and H. Then by conics will HLO be a tangent to the curve in the point L. Through A draw AI perpendicular to AC and conſequently a tangent to the curve, and LT perpendicular to LO. Now becauſe FC is equal to AC and FCA

FCA is a right angle, the angles FAC and AFC will each be half of a right angle. LOT will alſo be half of a right angle, becauſe LO is parallel to AF, and conſequently LTO is half of a right angle. If then the ellipſis repreſent a meridian of the earth IA and HO will repreſent the common ſections of that meridian and the horizons of two places; and AT, LT two perpendiculars to the horizons, and the angle ATL will be the difference of the latitude, (equal to 45°). But A is at the end of one of the axes of the ellipſis, and therefore the point L will repreſent a place in the latitude of 45°.

Since all the degrees of latitude increaſe in length as we approach to the pole, it is evident that the arch of 45° between the latitude of 45° and the pole, will be longer than the arch between the equator and the latitude of 45°. Now draw LS and LN parallel to BC and AC. Make BC $= a$, AC $= c$, LS $= x$, LN $= y$, LS $=$ NC, and LN $=$ CS. Then becauſe LOT is half of a right angle, and OSL is a right angle, OLS is alſo half of a right angle, therefore OS is equal to LS. In the ſame manner we prove HN equal to LN and conſequently HC equal to OC, put OC $(= x + y) = b$.

Then by conics $y : c :: c : b$ and $y = \frac{c^2}{b}$ and $b = \frac{c^2}{y}$

$$x : a :: a : b \text{ and } x = \frac{a^2}{b} \text{ and } b = \frac{a^2}{x}$$

Therefore $\frac{c^2}{y} = \frac{a^2}{x}$

and $y = \frac{c^2}{a^2} x$

but $b = \frac{a^2}{x} = x + y = \frac{a^2 + c^2}{a^2} x$

$$a^4 = \overline{a^2 + c^2} \times x^2$$

$x =$

$$x = \frac{a^3}{\sqrt{a^2 + c^2}}$$

$$y = \frac{c^2 x}{a^2} = \frac{c^2}{\sqrt{a^2 + c^2}}$$

$$b = \frac{a^2}{x} = \sqrt{a^2 + c^2}.$$

Put z = the length of the elliptic arch AL
v = that of BL

$\dot{z} = \frac{\dot{x}}{a}\sqrt{\frac{a^4 - a^2 x^2 + c^2 x^2}{a^2 - x^2}}$ by the nature of the curve: put $a^2 - c^2 = d^2$ then $\dot{z} = \frac{\dot{x}}{a}\sqrt{\frac{a^4 - d^2 x^2}{a^2 - x^2}} = \frac{\dot{x}}{a} \times \frac{\sqrt{a^4 - d^2 x^2}}{\sqrt{a^2 - x^2}} = \frac{\dot{x}}{a} \times \frac{\overline{a^4 - d^2 x^2}|^{\frac{1}{2}}}{\overline{a^2 - x^2}|^{\frac{1}{2}}}$. z = the fluent of $\frac{\dot{x}}{a} \times \frac{\overline{a^4 - d^2 x^2}|^{\frac{1}{2}}}{\overline{a^2 - x^2}|^{\frac{1}{2}}}$.

$$\overline{a^4 - d^2 x^2}|^{\frac{1}{2}} = a^2 - \frac{d^2 x^2}{2 a^2} - \frac{d^4 x^4}{8 a^6} - \frac{d^6 x^6}{16 a^{10}} - \frac{5 d^8 x^8}{128 a^{14}}, \text{ \&c.}$$

$$\overline{a^2 - x^2}|^{\frac{1}{2}} = a - \frac{x^2}{2 a} - \frac{x^4}{8 a^3} - \frac{x^6}{16 a^5} - \frac{5 x^8}{128 a^7}, \text{ \&c.}$$

The former of which ſeries being divided by the latter, the quotient is $a + \frac{c^2 x^2}{2 a^3} + \frac{c^2 x^4}{8 a^7} \times \overline{3 a^2 + d^2} + \frac{c^2 x^6}{16 a^{11}} \times \overline{5 a^4 + 2 a^2 d^2 + d^4} + \frac{c^2 x^8}{128 a^{15}} \times \overline{35 a^6 + 15 a^4 d^2 + 9 a^2 d^4 + 5 d^6}$, &c. which multiplied by $\frac{\dot{x}}{a}$ becomes

$$\dot{x} + \frac{c^2 x^2 \dot{x}}{2 a^4} + \frac{c^2 x^4 \dot{x}}{8 a^8} \times \overline{3 a^2 + d^2} + \frac{c^2 x^6 \dot{x}}{16 a^{12}} \times \overline{5 a^4 + 2 a^2 d^2 + d^4} + \frac{c^2 x^8 \dot{x}}{128 a^{16}} \times \overline{35 a^4 + 15 a^4 d^2 + 9 a^2 d^4 + 5 d^6}, \text{ \&c.}$$

the fluent of which is

$x +$

$$x+\frac{c^2x^3}{3.2a^4}+\frac{c^2x^5}{5.8a^8}\times\overline{3a^2+d^2}+\frac{c^2x^7}{7.16a^{12}}\times\overline{5a^4+2a^2d^2+d^4}$$
$$+\frac{c^2x^9}{9.128a^{16}}\times\overline{35a^6+15a^4d^2+9a^2d^4+5d^6},$$ &c. and by substituting for x its value $\frac{a^2}{\sqrt{a^2+c^2}}$ or $\frac{a^2}{b}$.

$$z=\frac{a^2}{b}+\frac{a^2c^2}{3.2b^3}+\frac{a^2c^2}{5.8b^5}\times\overline{3a^2+d^2}+\frac{a^2c^2}{7.16b^7}\times\overline{5a^4+2a^2d^2+d^4}$$
$$+\frac{a^2c^2}{9.128b^9}\times\overline{35a^6+15a^4d^2+9a^2d^4+5d^6},$$ &c. again

$$\dot{v}=\frac{\dot{y}}{c}\sqrt{\frac{c^4+d^2y^2}{c^2-y^2}}.$$

Which thrown into a series becomes,

$$\sqrt{c^4+d^2y^2}=c^2+\frac{d^2y^2}{2c^2}-\frac{d^4y^4}{8c^6}+\frac{d^6y^6}{16c^{10}}-\frac{5d^8y^8}{128c^{14}},$$ &c.

$$\sqrt{c^2-y^2}=c-\frac{y^2}{2c}-\frac{y^4}{8c^3}-\frac{y^6}{16c^5}-\frac{5y^8}{128c^7},$$ &c.

The former of which being divided by the latter becomes,

$$c+\frac{a^2y^2}{2c^3}+\frac{a^2y^4}{8c^7}\times\overline{3c^2-d^2}+\frac{a^2y^6}{16c^{11}}\times\overline{5c^4-2c^2d^2+d^4}$$
$$+\frac{a^2y^8}{128c^{15}}\times\overline{35c^6-15c^4d^2+9c^2d^4-5d^6},$$ &c. which

being multiplied by $\frac{\dot{y}}{c}$ is

$$\dot{v}=\dot{y}+\frac{a^2y^2\dot{y}}{2c^4}+\frac{a^2y^4\dot{y}}{8c^8}\times\overline{3c^2-d^2}+\frac{a^2y^6\dot{y}}{16c^{12}}\times$$
$$\overline{5c^4-2c^2d^2+d^4}+\frac{a^2y^8\dot{y}}{128c^{16}}\times\overline{35c^6-15c^4d^2+9c^2d^4-5d^6}$$

the fluent of which is

$$v=y+\frac{a^2y^3}{3.2c^4}+\frac{a^2y^5}{5.8c^8}\times\overline{3c^2-d^2}+\frac{a^2y^7}{7.16c^{12}}\times\overline{5c^4-2c^2d^2+d^4}$$

+

$+ \frac{a^2 y^9}{9.128\, c^{16}} \times \overline{35 c^6 - 15 c^4 d^2 + 9 c^2 d^4 - 5 d^6}$ and when $v = \frac{c^2}{b}$ the ſeries becomes

$$v = \frac{c^2}{b} + \frac{a^2 c^2}{3.2\, b^3} + \frac{a^2 c^2}{5.8\, b^5} \times \overline{3 c^2 - d^2} + \frac{a^2 c^2}{7.16\, b^7} \times \overline{5 c^4 - 2 c^2 d^2 + d^4} + \frac{a^2 c^2}{9.128\, b^9} \times \overline{35 c^6 - 15 c^4 d^2 + 9 c^2 d^4 - 5 d^6}$$

but

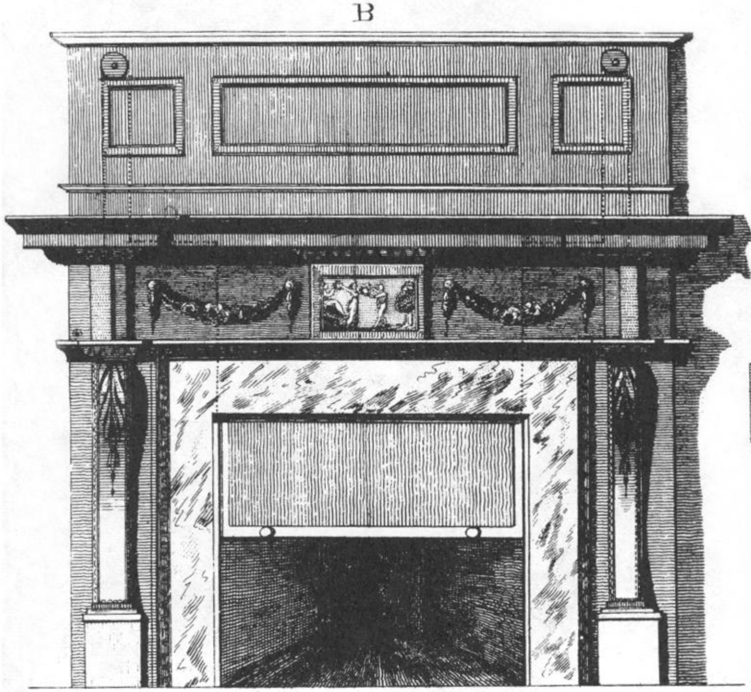
$$z = \frac{a^2}{b} + \frac{a^2 c^2}{3.2\, b^3} + \frac{a^2 c^2}{5.8\, b^5} \times \overline{3 a^2 + d^2} + \frac{a^2 c^2}{7.16\, b^7} \times \overline{5 a^4 + 2 a^2 d^2 + d^4} + \frac{a^2 c^2}{9.128\, b^9} \times \overline{35 a^6 + 15 a^4 d^2 + 9 a^2 d^4 + 5 d^6}.$$

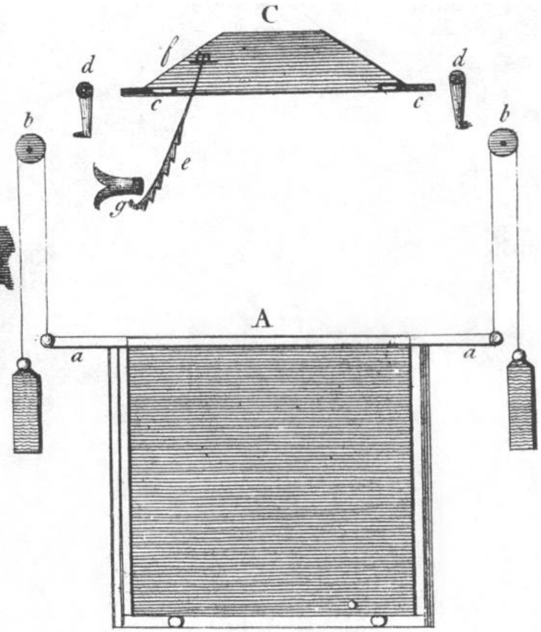
From a compariſon of theſe two equations, it will be ſeen that the law of continuation is the ſame in both, excepting that in the value of v, the ſigns of the odd powers of d^2 are negative, whereas in the value of z all the ſigns are affirmative. The powers and coefficients of a, c, and d, in the correſponding terms are the ſame; and to whatever number of terms the ſeries may be carried, it is evident that this will ſtill be the caſe. Hence if a be greater than c every term, except the ſecond, of the equation of the value of z, will be greater than the correſponding term of the equation of the value of v; conſequently the ſum of the ſeries $= z$ will be greater than the ſum of the ſeries $= v$: that is, if a be greater than c, z will be greater than v. Converſely if z be greater than v, a will be greater than c. If $a = c$, d^2 will vaniſh and the two ſeries will be equal to each other. If c be greater than a, d^2 will be negative, and the odd powers of d^2 in the ſeries $= z$, will in this caſe be negative, but in the ſeries $= v$ the odd powers of d^2 will become affirmative, and v will be greater than z; converſely if v be greater than z, c will be greater than a.

Hence,

Hence, if the arch AL exceed the arch LB, BC is greater than AC; but, if AD repreſent the axis of the earth, and BE the equatorial diameter, it is found by actual meaſurement, that each degree of the arch AL is greater than a degree of the arch BL, and conſequently the whole arch AL is greater than the whole arch BL, and therefore BC is greater than AC. Q. E. D.

Plate XIII.





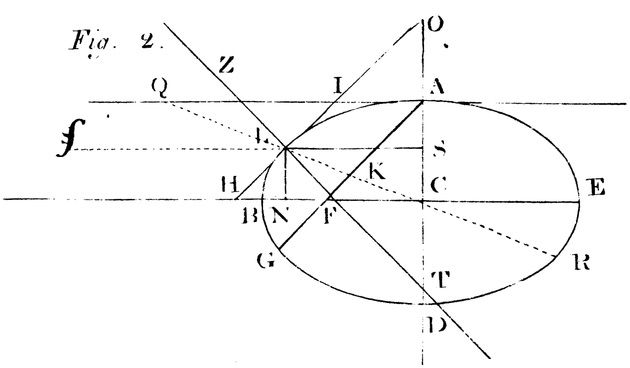


Fig. 2.

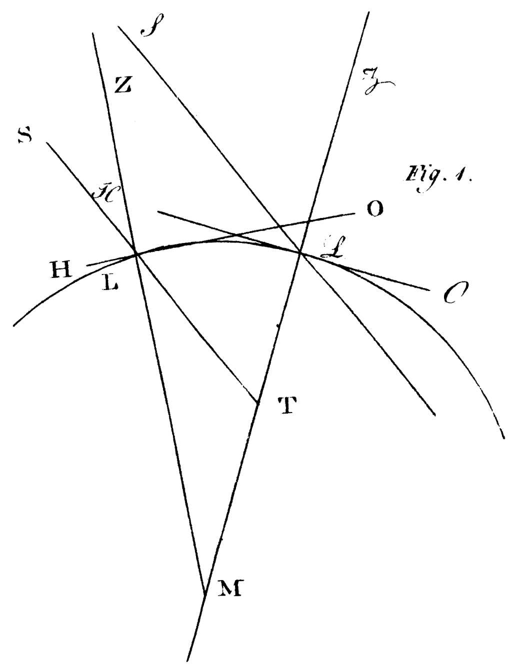


Fig. 1.

Weston Sc.